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TRANSITIVITY IN THE THEORY OF THE LORENTZ GROUP AND THE STOKES – MUELLER FORMALISM IN OPTICS

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Summary

Group-theoretical analysis of arbitrary polarization devices is performed, based on the theory of the Lorentz group. In effective "non-relativistic" Mueller case, described by 3-dimensional orthogonal matrices, results of the one polarization measurement $\mathbf{S} \xrightarrow{O} \mathbf{S}'$ determine group theoretical parameters within the accuracy of an arbitrary numerical variable. There are derived formulas, defining Muller parameter of the non-relativistic Mueller device uniquely and in explicit form by the results of two independent polarization measurements.

Analysis is extended to Lorentzian optical devices, described by 4-dimensional Mueller matrices. In this case, any single polarization measurement $(S_0, \mathbf{S}) \xrightarrow{L} (S'_0, \mathbf{S}')$ fixes parameters of the corresponding Mueller matrix up to 3 arbitrary variables. Formulas, defining Muller parameter of any relativistic Mueller device uniquely can be found from results of four independent polarization measurements. Analytical expressions for parameters of any Mueller device can be given the most simple form when using the results of 6 independent measurements, the corresponding formulas are written down in explicit form.

1. The transitivity problem in the theory of the Lorentz group

It is known that in describing (fully or partly) polarized light noticeable role may given to the group of 3 + 1-pseudoorthogonal transformations consisting of a group $SO(3, 1)$ isomorphic to the Lorentz group. Therefore, techniques developed in the frames of the Lorentz group, in particular within relativistic kinematics, may play heuristic role in exploring optical problems (see big list of references in the end; a previous consideration of one of the authors is given in [101].

In the paper, when working with the Lorentz group we use technique developed in [102] and [103] and partly updated in [104]. This approach had been started many years ago by Einstein and Mayer in [105].

Let us recall the known transitivity problem in relativistic kinematics: in Stokes – Mueller approach it reads

$$L_b^a(k, \bar{k}^*) S_a = +S'_b. \quad (1)$$

From the very beginning, one peculiarity should be noted: due to existence of the concept of little Lorentz group initial and final Stokes 4-vectors S and S' , one can write down the transitivity condition in the form $L(L_{\text{little}} S) = L'_{\text{little}} S'$, so that

$$[(L'_{\text{little}})^{-1} L L_{\text{little}}] S = S'. \quad (2)$$

This means that the transitive matrix L cannot be defined uniquely in terms of S and S' .

Let us use the factorized representation for Lorentzian matrices (we adhere notation given in [101, 104]), eq. (1) gives

$$A^* S = A^{-1} S' , \quad \text{and} \quad A S = (A^*)^{-1} S' , \quad (3)$$

or in more detailed form (conjugate equation is written down too)

$$\begin{aligned} \left| \begin{array}{cccc} k_0^* & -k_1^* & -k_2^* & -k_3^* \\ -k_1^* & k_0^* & ik_3^* & -ik_2^* \\ -k_2^* & -ik_3^* & k_0^* & ik_1^* \\ -k_3^* & ik_2^* & -ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| &= \left| \begin{array}{cccc} k_0 & k_1 & k_2 & k_3 \\ k_1 & k_0 & ik_3 & -ik_2 \\ k_2 & -ik_3 & k_0 & ik_1 \\ k_3 & ik_2 & -ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| , \\ \left| \begin{array}{cccc} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| &= \left| \begin{array}{cccc} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| . \end{aligned} \quad (4)$$

Below, the notation will be used

$$k_0 = n_0 + im_0 , \quad k_j = -in_j + m_j , \quad k_0 - \mathbf{k}^2 = 1 .$$

Summing and subtracting eqs we get

$$\begin{aligned} \left| \begin{array}{cccc} n_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & n_0 & -n_3 & n_2 \\ -m_2 & n_3 & n_0 & -n_1 \\ -m_3 & -n_2 & n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| &= \left| \begin{array}{cccc} n_0 & m_1 & m_2 & m_3 \\ m_1 & n_0 & n_3 & -n_2 \\ m_2 & -n_3 & n_0 & n_1 \\ m_3 & n_2 & -n_1 & n_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| , \\ \left| \begin{array}{cccc} -m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & -m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & -m_0 & m_1 \\ -n_3 & m_2 & -m_1 & -m_0 \end{array} \right| \left| \begin{array}{c} S_0 \\ S_1 \\ S_2 \\ S_3 \end{array} \right| &= \left| \begin{array}{cccc} m_0 & -n_1 & -n_2 & -n_3 \\ -n_1 & m_0 & m_3 & -m_2 \\ -n_2 & -m_3 & m_0 & m_1 \\ -n_3 & m_2 & -m_1 & m_0 \end{array} \right| \left| \begin{array}{c} S'_0 \\ S'_1 \\ S'_2 \\ S'_3 \end{array} \right| . \end{aligned}$$

So, we arrive at two homogeneous linear systems under 8 variances

$$\begin{aligned} n_0 (S_0 - S'_0) - m_1 (S_1 + S'_1) - m_2 (S_2 + S'_2) - m_3 (S_3 + S'_3) &= 0 , \\ -m_1 (S_0 + S'_0) + n_0 (S_1 - S'_1) + n_2 (S_3 + S'_3) - n_3 (S_2 + S'_2) &= 0 , \\ -m_2 (S_0 + S'_0) + n_0 (S_2 - S'_2) + n_3 (S_1 + S'_1) - n_1 (S_3 + S'_3) &= 0 , \\ -m_3 (S_0 + S'_0) + n_0 (S_3 - S'_3) + n_1 (S_2 + S'_2) - n_2 (S_1 + S'_1) &= 0 , \\ -m_0 (S_0 + S'_0) - n_1 (S_1 - S'_1) - n_2 (S_2 - S'_2) - n_3 (S_3 - S'_3) &= 0 , \\ -n_1 (S_0 - S'_0) - m_0 (S_1 + S'_1) - m_2 (S_3 - S'_3) + m_3 (S_2 - S'_2) &= 0 , \\ -n_2 (S_0 - S'_0) - m_0 (S_2 + S'_2) - m_3 (S_1 - S'_1) + m_1 (S_3 - S'_3) &= 0 , \\ -n_3 (S_0 - S'_0) - m_0 (S_3 + S'_3) - m_1 (S_2 - S'_2) + m_2 (S_1 - S'_1) &= 0 . \end{aligned} \quad (5)$$

2. "Non-relativistic" 3-dimensional Mueller matrices

First, let us consider more simple (non-relativistic) case when $S'_0 = S_0 = I = \text{inv.}$ Eqs. (5) takes the form (because we search solutions in 3-dimensional rotations, we require $m_0 = 0, m_j = 0$):

$$\begin{aligned} n_0 (S_1 - S'_1) + n_2 (S_3 + S'_3) - n_3 (S_2 + S'_2) &= 0 , \\ n_0 (S_2 - S'_2) + n_3 (S_1 + S'_1) - n_1 (S_3 + S'_3) &= 0 , \\ n_0 (S_3 - S'_3) + n_1 (S_2 + S'_2) - n_2 (S_1 + S'_1) &= 0 , \\ -n_1 (S_1 - S'_1) - n_2 (S_2 - S'_2) - n_3 (S_3 - S'_3) &= 0 . \end{aligned} \quad (6)$$

The fourth equation is not independent of three remaining – it follows from them. Therefore we have the system of 3 independent ones

$$\begin{aligned} n_2 (S_3 + S'_3) - n_3 (S_2 + S'_2) &= -n_0 (S_1 - S'_1) , \\ n_3 (S_1 + S'_1) - n_1 (S_3 + S'_3) &= -n_0 (S_2 - S'_2) , \\ n_1 (S_2 + S'_2) - n_2 (S_1 + S'_1) &= -n_0 (S_3 - S'_3) . \end{aligned} \quad (7)$$

They may be written in 3-vector form

$$\mathbf{n} \times (\mathbf{S} + \mathbf{S}') = -n_0 (\mathbf{S} - \mathbf{S}') . \quad (8)$$

General solutions for \mathbf{n} can be searched with the aid of substitution

$$\mathbf{n} = \alpha \mathbf{S} + \rho \mathbf{S}' + \beta \mathbf{S} \times \mathbf{S}' ,$$

then eq. (8) leads to (below note $S^2 = \mathbf{S}\mathbf{S}$)

$$(\alpha - \rho) \mathbf{S} \times \mathbf{S}' + \beta [\mathbf{S}' S^2 + \mathbf{S}' (\mathbf{S}\mathbf{S}') - \mathbf{S} S^2 - \mathbf{S} (\mathbf{S}\mathbf{S}')] = -n_0 \mathbf{S} + n_0 \mathbf{S}' ,$$

from whence it follow $\rho = \alpha$, α is arbitrary, and

$$n_0 = \beta (S^2 + \mathbf{S} \mathbf{S}') , \quad \mathbf{n} = \alpha (\mathbf{S} + \mathbf{S}') + \beta \mathbf{S} \times \mathbf{S}' . \quad (9)$$

One must to take into account additional restriction for parameters of rotation matrices

$$n_0^2 + \mathbf{n}^2 = 1 , \quad (10)$$

which results in

$$\beta^2 (S^2 + \mathbf{S} \mathbf{S}')^2 + [\alpha (\mathbf{S} + \mathbf{S}') + \beta \mathbf{S} \times \mathbf{S}']^2 = 1 ,$$

or

$$\beta^2 [S^4 + 2S^2 (\mathbf{S} \mathbf{S}') + (\mathbf{S} \mathbf{S}')^2] + \beta^2 [S^4 - (\mathbf{S}\mathbf{S}')^2] + 2\alpha^2 (S^2 + \mathbf{S} \mathbf{S}') = 1 ;$$

and ultimately eq. (10) gives

$$\beta^2 S^2 + \alpha^2 = \frac{1}{2(S^2 + \mathbf{S} \mathbf{S}')} . \quad (11)$$

General solution of eq. (11) can be presented in terms of sin- and cos-functions of an angular variable

$$\alpha = \frac{\sin \Gamma}{\sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} , \quad \beta = \frac{\cos \Gamma}{S \sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} , \quad \Gamma \in [0, 2\pi] . \quad (12)$$

Thus, relations (9) read (here $\Gamma \in [0, 2\pi]$ stands for arbitrary parameter)

$$\begin{aligned} n_0^2 + \mathbf{n}^2 &= 1 , \quad n_0 = \frac{\cos \Gamma}{S \sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} (S^2 + \mathbf{S} \mathbf{S}') , \\ \mathbf{n} &= \frac{\sin \Gamma}{\sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} (\mathbf{S} + \mathbf{S}') + \frac{\cos \Gamma}{S \sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} \mathbf{S} \times \mathbf{S}' . \end{aligned} \quad (13)$$

Note that when $\mathbf{S}' = \mathbf{S}$, relations (13) describe the case of little rotation group

$$n_0^2 + \mathbf{n}^2 = 1 , \quad n_0 = \cos \Gamma , \quad \mathbf{n} = \sin \Gamma \frac{\mathbf{S}}{S} . \quad (14)$$

When $\Gamma = 0$, solution (13) becomes of the most simple form

$$n_0 = \frac{S^2 + \mathbf{S} \mathbf{S}'}{S \sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} , \quad \mathbf{n} = \frac{\mathbf{S} \times \mathbf{S}'}{S \sqrt{2(S^2 + \mathbf{S} \mathbf{S}')}} . \quad (15)$$

Note, that we may transform all the relations to a Gibbs 3-vector parameter in the rotation group (the full treatment of the theory in this parametrization see in [102])

$$\mathbf{c} = \frac{\mathbf{n}}{n_0} , \quad (16)$$

then eqs. (13) give

$$\mathbf{c} = \operatorname{tg} \Gamma \frac{S}{S^2 + \mathbf{S} \mathbf{S}'} (\mathbf{S} + \mathbf{S}') + \frac{\mathbf{S} \times \mathbf{S}'}{S^2 + \mathbf{S} \mathbf{S}'} . \quad (17)$$

Note that in the non-relativistic case, for Stokes vectors one can use the following parametrization (I is intensity of the light beam, p is a polarization degree)

$$S_0 = I, \quad \mathbf{S} = Ip \mathbf{N}, \quad I - \operatorname{inv} , \quad \mathbf{N}^2 = 1 ; \quad (18)$$

at this (13) and (15) change to

$$\begin{aligned} n_0^2 + \mathbf{n}^2 &= 1 , \quad n_0 = \cos \Gamma \frac{1 + \mathbf{N} \mathbf{N}'}{\sqrt{2(1 + \mathbf{N} \mathbf{N}')}} , \\ \mathbf{n} &= \sin \Gamma \frac{\mathbf{N} + \mathbf{N}'}{\sqrt{2(1 + \mathbf{N} \mathbf{N}')}} + \cos \Gamma \frac{\mathbf{N} \times \mathbf{N}'}{\sqrt{2(1 + \mathbf{N} \mathbf{N}')}} , \end{aligned} \quad (19)$$

and

$$\mathbf{c} = \operatorname{tg} \Gamma \frac{\mathbf{N} + \mathbf{N}'}{1 + \mathbf{N} \mathbf{N}'} + \frac{\mathbf{N} \times \mathbf{N}'}{1 + \mathbf{N} \mathbf{N}'} . \quad (20)$$

3. On defining Mueller 3-matrices from the results of polarization measurements

Because a single polarization measurement relating $\mathbf{S} \xrightarrow{L} \mathbf{S}'_1$ cannot fix Mueller 3-matrix uniquely, to obtain result values for parameters of the Mueller 3-matrix, one need to perform two independent measurements $\mathbf{S}_1 \xrightarrow{L} \mathbf{S}'_1$, $\mathbf{S}_2 \xrightarrow{L} \mathbf{S}'_2$. Mathematically, the problem of finding a definite Mueller 3-matrix can be formulated as a system to solve, describing two polarization measurement with one the same Mueller matrix.

First, let us consider this task with the aid of Gibbs 3-paramere

$$\mathbf{c} = \text{tg } \Gamma \frac{\mathbf{N}_1 + \mathbf{N}'_1}{1 + \mathbf{N}_1 \mathbf{N}'_1} + \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1 \mathbf{N}'_1}, \quad \mathbf{c} = \text{tg } \Gamma \frac{\mathbf{N}_2 + \mathbf{N}'_2}{1 + \mathbf{N}_2 \mathbf{N}'_2} + \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{1 + \mathbf{N}_2 \mathbf{N}'_2}; \quad (21)$$

so we have a vector equation

$$\text{tg } \Gamma \left[\frac{\mathbf{N}_1 + \mathbf{N}'_1}{1 + \mathbf{N}_1 \mathbf{N}'_1} - \frac{\mathbf{N}_2 + \mathbf{N}'_2}{1 + \mathbf{N}_2 \mathbf{N}'_2} \right] + \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1 \mathbf{N}'_1} - \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{1 + \mathbf{N}_2 \mathbf{N}'_2} = 0. \quad (22)$$

Multiplying it by $\mathbf{N}_1, \mathbf{N}'_1, \mathbf{N}_2, \mathbf{N}'_2$, we obtain four scalar equations

$$\begin{aligned} \text{tg } \Gamma \left[1 - \frac{\mathbf{N}_1(\mathbf{N}_2 + \mathbf{N}'_2)}{1 + \mathbf{N}_2 \mathbf{N}'_2} \right] - \frac{\mathbf{N}_1(\mathbf{N}_2 \times \mathbf{N}'_2)}{1 + \mathbf{N}_2 \mathbf{N}'_2} &= 0, \\ \text{tg } \Gamma \left[1 - \frac{\mathbf{N}'_1(\mathbf{N}_2 + \mathbf{N}'_2)}{1 + \mathbf{N}_2 \mathbf{N}'_2} \right] - \frac{\mathbf{N}'_1(\mathbf{N}_2 \times \mathbf{N}'_2)}{1 + \mathbf{N}_2 \mathbf{N}'_2} &= 0, \\ \text{tg } \Gamma \left[\frac{\mathbf{N}_2(\mathbf{N}_1 + \mathbf{N}'_1)}{1 + \mathbf{N}_1 \mathbf{N}'_1} - 1 \right] + \frac{\mathbf{N}_2(\mathbf{N}_1 \times \mathbf{N}'_1)}{1 + \mathbf{N}_1 \mathbf{N}'_1} &= 0, \\ \text{tg } \Gamma \left[\frac{\mathbf{N}'_2(\mathbf{N}_1 + \mathbf{N}'_1)}{1 + \mathbf{N}_1 \mathbf{N}'_1} - 1 \right] + \frac{\mathbf{N}'_2 \mathbf{N}_1 \times \mathbf{N}'_1}{1 + \mathbf{N}_1 \mathbf{N}'_1} &= 0. \end{aligned} \quad (23)$$

From whence it follow

$$\begin{aligned} \text{tg } \Gamma &= \frac{\mathbf{N}_1 (\mathbf{N}_2 \times \mathbf{N}'_2)}{(\mathbf{N}_2 - \mathbf{N}_1)(\mathbf{N}_2 + \mathbf{N}'_2)}, & \text{tg } \Gamma &= -\frac{\mathbf{N}'_1 (\mathbf{N}'_2 \times \mathbf{N}_2)}{(\mathbf{N}'_2 - \mathbf{N}'_1)(\mathbf{N}'_2 + \mathbf{N}_2)}, \\ \text{tg } \Gamma &= \frac{\mathbf{N}_2 (\mathbf{N}_1 \times \mathbf{N}'_1)}{(\mathbf{N}_1 - \mathbf{N}_2)(\mathbf{N}_1 + \mathbf{N}'_1)}, & \text{tg } \Gamma &= -\frac{\mathbf{N}'_2 (\mathbf{N}'_1 \times \mathbf{N}_1)}{(\mathbf{N}'_1 - \mathbf{N}'_2)(\mathbf{N}'_1 + \mathbf{N}_1)}. \end{aligned} \quad (24)$$

Thus, we have a simple expression for $\text{tg } \Gamma$, together with four additional constraints, which determine the whole aggregate of all possible couples of Stokes 3-vectors related by one the same Mueller matrices.

Now let us detail considering of the task in the frames of unitary group $SU(2)$ – evidently, two solutions cannot contradict each other. Here we have

$$\begin{aligned} n_0 &= \beta_1 \mathbf{S}_1(\mathbf{S}_1 + \mathbf{S}'_1), & \mathbf{n} &= \alpha_1 (\mathbf{S}_1 + \mathbf{S}'_1) + \beta_1 \mathbf{S}_1 \times \mathbf{S}'_1, \\ n_0 &= \beta_2 \mathbf{S}_2(\mathbf{S}_2 + \mathbf{S}'_2), & \mathbf{n} &= \alpha_2 (\mathbf{S}_2 + \mathbf{S}'_2) + \beta_2 \mathbf{S}_2 \times \mathbf{S}'_2. \end{aligned} \quad (25)$$

what is equivalent to

$$\begin{aligned} n_0 &= \cos \Gamma \frac{1 + \mathbf{N}_1 \mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1 \mathbf{N}'_1)}}, & \mathbf{n} &= \sin \Gamma \frac{\mathbf{N}_1 + \mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1 \mathbf{N}'_1)}} + \cos \Gamma \frac{\mathbf{N}_1 \times \mathbf{N}'_1}{\sqrt{2(1 + \mathbf{N}_1 \mathbf{N}'_1)}} \\ n_0 &= \cos \Gamma \frac{1 + \mathbf{N}_2 \mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2 \mathbf{N}'_2)}}, & \mathbf{n} &= \sin \Gamma \frac{\mathbf{N}_2 + \mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2 \mathbf{N}'_2)}} + \cos \Gamma \frac{\mathbf{N}_2 \times \mathbf{N}'_2}{\sqrt{2(1 + \mathbf{N}_2 \mathbf{N}'_2)}} \end{aligned} \quad (26)$$

From two different expressions for n_0 , it follows

$$\mathbf{N}_1 \mathbf{N}'_1 = \mathbf{N}_2 \mathbf{N}'_2 . \quad (27)$$

Taking this into account, from two different expressions for \mathbf{n} we derive

$$\sin \Gamma [(\mathbf{N}_1 + \mathbf{N}'_1) - (\mathbf{N}_2 + \mathbf{N}'_2)] + \cos \Gamma [(\mathbf{N}_1 \times \mathbf{N}'_1) - (\mathbf{N}_2 \times \mathbf{N}'_2)] = 0 \quad (28)$$

It should be noted that due to (27), relation (22) becomes much more simpler

$$\text{tg } \Gamma [(\mathbf{N}_1 + \mathbf{N}'_1) - (\mathbf{N}_2 + \mathbf{N}'_2)] + \mathbf{N}_1 \times \mathbf{N}'_1 - \mathbf{N}_2 \times \mathbf{N}'_2 = 0 . \quad (29)$$

In fact, (28) and (29) coincide, difference consist in the following: (28) cannot distinguish between two solutions: $(+\cos \Gamma, +\sin \Gamma)$ and $(-\cos \Gamma, -\sin \Gamma)$.

4. Relativistic Mueller matrices relating two Stokes 4-vectors

Let us turn back to general (relativistic) case of Mueller matrices (5):

$$\begin{aligned} m_1 (S_1 + S'_1) + m_2 (S_2 + S'_2) + m_3 (S_3 + S'_3) &= n_0 (S_0 - S'_0) , \\ m_1 (S_0 + S'_0) - n_2 (S_3 + S'_3) + n_3 (S_2 + S'_2) &= n_0 (S_1 - S'_1) , \\ m_2 (S_0 + S'_0) - n_3 (S_1 + S'_1) + n_1 (S_3 + S'_3) &= n_0 (S_2 - S'_2) , \\ m_3 (S_0 + S'_0) - n_1 (S_2 + S'_2) + n_2 (S_1 + S'_1) &= n_0 (S_3 - S'_3) , \\ -n_1 (S_1 - S'_1) - n_2 (S_2 - S'_2) - n_3 (S_3 - S'_3) &= m_0 (S_0 + S'_0) , \\ -n_1 (S_0 - S'_0) - m_2 (S_3 - S'_3) + m_3 (S_2 - S'_2) &= m_0 (S_1 + S'_1) , \\ -n_2 (S_0 - S'_0) - m_3 (S_1 - S'_1) + m_1 (S_3 - S'_3) &= m_0 (S_2 + S'_2) , \\ -n_3 (S_0 - S'_0) - m_1 (S_2 - S'_2) + m_2 (S_1 - S'_1) &= m_0 (S_3 + S'_3) . \end{aligned} \quad (30)$$

Because we search solutions among proper orthochronous Lorentzian transformations, unknown parameters must obey additional relations

$$n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1 , \quad n_0 m_0 + \mathbf{n} \mathbf{m} = 0 ; \quad (31)$$

by this reason, the trivial solution $n_a = 0, m_a = 0$ for (30) is of no interest. Eqs. (30) can be rewritten in 3-vector form

$$\begin{aligned} \mathbf{m} (\mathbf{S} + \mathbf{S}') &= n_0 (S_0 - S'_0) , \\ \mathbf{n} (\mathbf{S} - \mathbf{S}') &= -m_0 (S_0 + S'_0) , \\ \mathbf{m} (S_0 + S'_0) + (\mathbf{S} + \mathbf{S}') \times \mathbf{n} &= n_0 (\mathbf{S} - \mathbf{S}') , \\ \mathbf{n} (S_0 - S'_0) - (\mathbf{S} - \mathbf{S}') \times \mathbf{m} &= -m_0 (\mathbf{S} + \mathbf{S}') . \end{aligned} \quad (32)$$

Note that the (non-relativity) requirement $S_0 - S'_0 = 0$ immediately leads us to additional relations $\mathbf{m} = 0$ and $m_0 = 0$, and we get eqs. (7)–(8).

Let us introduce notation

$$\begin{aligned} S_0 + S'_0 &= A , \quad S_0 - S'_0 = B , \quad \mathbf{S} + \mathbf{S}' = \mathbf{A} , \\ \mathbf{S} - \mathbf{S}' &= \mathbf{B} , \quad N_+ = \nu , \quad M_- = \mu ; \end{aligned} \quad (33)$$

The complete system of equations to solve is

$$n_0^2 + \mathbf{n}^2 - m_0^2 - \mathbf{m}^2 = 1, \quad n_0 m_0 + \mathbf{n} \mathbf{m} = 0; \quad (34)$$

$$\mathbf{m} \mathbf{A} = n_0 \mathbf{B}, \quad \mathbf{n} \mathbf{B} = -m_0 \mathbf{A}; \quad (35)$$

$$\mathbf{m} \mathbf{A} + \mathbf{A} \times \mathbf{n} = n_0 \mathbf{B}, \quad \mathbf{n} \mathbf{B} - \mathbf{B} \times \mathbf{m} = -m_0 \mathbf{A}. \quad (36)$$

It is convenient to use linear expansions for both 3-vectors

$$\mathbf{n} = N_+ \mathbf{A} + N_- \mathbf{B} + N \mathbf{A} \times \mathbf{B}, \quad \mathbf{m} = M_+ \mathbf{A} + M_- \mathbf{B} + M \mathbf{A} \times \mathbf{B}. \quad (37)$$

From the first equation in (36) it follows

$$A(M_+ \mathbf{A} + M_- \mathbf{B} + M \mathbf{A} \times \mathbf{B}) + \mathbf{A} \times (N_- \mathbf{B} + N \mathbf{A} \times \mathbf{B}) = n_0 \mathbf{B},$$

which gives three equations

$$AM_+ + \mathbf{A} \mathbf{B} N = 0, \quad AM_- - \mathbf{A}^2 N = n_0, \quad AM + N_- = 0. \quad (38)$$

In the same manner, from the second equation in (36) we get

$$B(N_+ \mathbf{A} + N_- \mathbf{B} + N \mathbf{A} \times \mathbf{B}) - \mathbf{B} \times (M_+ \mathbf{A} + M \mathbf{A} \times \mathbf{B}) = -m_0 \mathbf{A},$$

and further

$$BN_- + \mathbf{A} \mathbf{B} M = 0, \quad BN_+ - \mathbf{B}^2 M = -m_0, \quad BN + M_+ = 0. \quad (39)$$

Thus, two vector equations (36) provide us with the system for six parameters

$$\begin{aligned} AM_+ + \mathbf{A} \mathbf{B} N &= 0, & AM_- - \mathbf{A}^2 N &= n_0, & AM + N_- &= 0; \\ BN_- + \mathbf{A} \mathbf{B} M &= 0, & BN_+ - \mathbf{B}^2 M &= -m_0, & BN + M_+ &= 0. \end{aligned} \quad (40)$$

After excluding the variables N_-, M_+ :

$$N_- = -AM, \quad M_+ = -BN, \quad (41)$$

eqs. (40) read

$$\begin{aligned} -ABN + \mathbf{A} \mathbf{B} N &= 0, & AM_- - \mathbf{A}^2 N &= n_0, \\ -ABM + \mathbf{A} \mathbf{B} M &= 0, & BN_+ - \mathbf{B}^2 M &= -m_0. \end{aligned} \quad (42)$$

Note that equations 1 and 3 are identities. In fact, eqs. (42) are equivalent to two equations only

$$AM_- - \mathbf{A}^2 N = n_0, \quad BN_+ - \mathbf{B}^2 M = -m_0, \quad (43)$$

Substituting expressions

$$\mathbf{n} = N_+ \mathbf{A} - M \mathbf{A} \mathbf{B} + N \mathbf{A} \times \mathbf{B}, \quad \mathbf{m} = M_- \mathbf{B} - N \mathbf{B} \mathbf{A} + M \mathbf{A} \times \mathbf{B}; \quad (44)$$

into (35), we arrive at

$$\begin{aligned} M_- \mathbf{B} \mathbf{A} - N B \mathbf{A}^2 &= n_0 B \implies M_- A - N \mathbf{A}^2 = n_0 , \\ N_+ \mathbf{A} \mathbf{B} - M A \mathbf{B}^2 &= -m_0 A \implies N_+ B - M \mathbf{B}^2 = -m_0 ; \end{aligned}$$

which coincide with (43). This means that eqs. (35) can be removed. The above substitutions for two vectors (44) are to be allowed in the conditions

$$n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2 , \quad n_0 m_0 = -\mathbf{n} \mathbf{m} = 0 .$$

Let us simplify notation

$$M_- = x , \quad N = y , \quad N_+ = z , \quad M = w$$

In these variables, the main equations to solve read

$$\begin{aligned} n_0 &= A x - \mathbf{A}^2 y , \quad \mathbf{n} = z \mathbf{A} - w A \mathbf{B} + y \mathbf{A} \times \mathbf{B} ; , \\ m_0 &= -B z + \mathbf{B}^2 w , \quad \mathbf{m} = x \mathbf{B} - y B \mathbf{A} + w \mathbf{A} \times \mathbf{B} ; \\ n_0 m_0 &= -\mathbf{n} \mathbf{m} , \quad n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2 . \end{aligned} \tag{45}$$

First, let us detail $n_0 m_0 = -\mathbf{n} \mathbf{m}$. Taking into account

$$\begin{aligned} n_0 m_0 &= -xz AB + wx A \mathbf{B}^2 + yz B \mathbf{A}^2 - wy \mathbf{A}^2 \mathbf{B}^2 , \\ -\mathbf{n} \mathbf{m} &= -(z \mathbf{A} - w A \mathbf{B} + y \mathbf{A} \times \mathbf{B}) (x \mathbf{B} - y B \mathbf{A} + w \mathbf{A} \times \mathbf{B}) = \\ &= -xz \mathbf{A} \mathbf{B} + yz B \mathbf{A}^2 + wx A \mathbf{B}^2 - yw A B \mathbf{A} \mathbf{B} - yw \mathbf{A}^2 \mathbf{B}^2 + yw (\mathbf{A} \mathbf{B})^2 . \end{aligned}$$

we arrive at

$$0 = xz (AB - \mathbf{A} \mathbf{B}) - yw A B \mathbf{A} \mathbf{B} + yw (\mathbf{A} \mathbf{B})^2 . \tag{46}$$

Because

$$AB - \mathbf{A} \mathbf{B} = (S_0^2 - \mathbf{S}^2) - (S_0'^2 - \mathbf{S}'^2) = 0 , \tag{47}$$

eq. (46) takes the form of an identity $0 = 0$, subsequently, this equation can be excluded from (45). Remaining and independent relations are

$$\begin{aligned} n_0^2 - m_0^2 &= 1 + \mathbf{m}^2 - \mathbf{n}^2 , \\ n_0 &= A x - \mathbf{A}^2 y , \quad \mathbf{n} = z \mathbf{A} - w A \mathbf{B} + y \mathbf{A} \times \mathbf{B} , \\ m_0 &= -B z + \mathbf{B}^2 w , \quad \mathbf{m} = x \mathbf{B} - y B \mathbf{A} + w \mathbf{A} \times \mathbf{B} . \end{aligned} \tag{48}$$

Each of vector equation in (48) can be changed into three scalar ones; those are obtained through multiplying them by $\mathbf{A}, \mathbf{B}, \mathbf{A} \times \mathbf{B}$:

$$\begin{aligned} \mathbf{A} \mathbf{n} &= z \mathbf{A}^2 - w A^2 B , \\ \mathbf{B} \mathbf{n} &= z AB - w A \mathbf{B}^2 , \\ (\mathbf{A} \times \mathbf{B}) \mathbf{n} &= +y \mathbf{A}^2 \mathbf{B}^2 - y A^2 B^2 , \\ \mathbf{A} \mathbf{m} &= x AB - y B \mathbf{A}^2 , \\ \mathbf{B} \mathbf{m} &= x \mathbf{B}^2 - y B^2 A , \\ (\mathbf{A} \times \mathbf{B}) \mathbf{m} &= +w \mathbf{A}^2 \mathbf{B}^2 - w A^2 B^2 . \end{aligned} \tag{49}$$

These equations are easy to solve

$$\begin{aligned} y &= \frac{(\mathbf{A} \times \mathbf{B})\mathbf{n}}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}, \quad z = -\frac{(\mathbf{Bn})AB - (\mathbf{An})\mathbf{B}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}, \quad w = -\frac{1}{A} \frac{(\mathbf{Bn})\mathbf{A}^2 - (\mathbf{An})AB}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}; \\ w &= \frac{(\mathbf{A} \times \mathbf{B})\mathbf{m}}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}, \quad x = \frac{-(\mathbf{Am})AB + (\mathbf{Bm})\mathbf{A}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}, \quad y = \frac{1}{B} \frac{(\mathbf{Bm})AB - (\mathbf{Am})\mathbf{B}^2}{\mathbf{A}^2\mathbf{B}^2 - A^2B^2}. \end{aligned} \quad (50)$$

Taking (48), we may turn back to a starting complex parameter k_a :

$$\begin{aligned} k_0 &= n_0 + im_0 = (xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2), \\ \mathbf{k} &= \mathbf{m} - i\mathbf{n} = -(yB + iz)\mathbf{A} + (x + iwA)\mathbf{B} + (w - iy)\mathbf{A} \times \mathbf{B}. \end{aligned} \quad (51)$$

Note that one can derive a more simple 3-vector, parameter for Lorentz group [...],

$$\mathbf{q} = \frac{\mathbf{k}}{k_0} = \frac{-(yB + iz)\mathbf{A} + (x + iwA)\mathbf{B} + (w - iy)\mathbf{A} \times \mathbf{B}}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)} \quad (52)$$

It may be formally simplified

$$\begin{aligned} \mathbf{q} &= \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{A} \times \mathbf{B}, \\ \alpha &= \frac{-(yB + iz)}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}, \\ \beta &= \frac{x + iwA}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}, \\ \gamma &= \frac{w - iy}{(xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2)}. \end{aligned} \quad (53)$$

The formulas allow transition to a more simple non-relativistic case ($x \equiv 0$, $w \equiv 0$, $B = 0$)

$$\begin{aligned} \mathbf{c} &= i\mathbf{q} = i\alpha \mathbf{A} + i\beta \mathbf{B} + i\gamma \mathbf{A} \times \mathbf{B}, \\ i\alpha &= -\frac{1}{\mathbf{A}^2} \frac{z}{y}, \quad i\beta = 0, \quad i\gamma = -\frac{1}{\mathbf{A}^2}; \end{aligned} \quad (54)$$

these relations describe 1-parametric set of 3-rotations. In relations (48), the non-relativistic case is reached as follow

$$n_0^2 + \mathbf{n}^2 = 1, \quad n_0 = y\mathbf{A}^2, \quad \mathbf{n} = z\mathbf{A} + y\mathbf{A} \times \mathbf{B}. \quad (55)$$

let us obtain an explicit form of the relationship $n_0^2 - m_0^2 = 1 + \mathbf{m}^2 - \mathbf{n}^2$ in (48). We have

$$\begin{aligned} n_0^2 - m_0^2 &= (Ax - \mathbf{A}^2y)^2 - (-Bz + \mathbf{B}^2w)^2 = \\ &= A^2x^2 - B^2z^2 - 2AA^2xy + 2BB^2zw + (\mathbf{A}^2)^2y^2 - (\mathbf{B}^2)^2w^2, \end{aligned}$$

and further

$$\begin{aligned} \mathbf{m}^2 &= (x\mathbf{B} - yB\mathbf{A} + w\mathbf{A} \times \mathbf{B})(x\mathbf{B} - yB\mathbf{A} + w\mathbf{A} \times \mathbf{B}) = \\ &= x^2\mathbf{B}^2 - xyB(\mathbf{BA}) - xyB(\mathbf{BA}) + y^2B^2\mathbf{A}^2 + w^2\mathbf{A}^2\mathbf{B}^2 - w^2(\mathbf{AB})^2, \end{aligned}$$

that is

$$\mathbf{m}^2 = x^2 \mathbf{B}^2 - 2xy AB^2 + y^2 B^2 \mathbf{A}^2 + w^2 \mathbf{A}^2 \mathbf{B}^2 - w^2 A^2 B^2 .$$

In the same manner, we derive

$$\begin{aligned} \mathbf{n}^2 &= (z\mathbf{A} - wAB + y\mathbf{A} \times \mathbf{B}) (z\mathbf{A} - wAB + y\mathbf{A} \times \mathbf{B}) = \\ &= z^2 \mathbf{A}^2 - 2zw BA^2 + w^2 A^2 \mathbf{B}^2 + y^2 \mathbf{A}^2 \mathbf{B}^2 - y^2 A^2 B^2 , \end{aligned}$$

and further

$$\begin{aligned} 1 + \mathbf{m}^2 - \mathbf{n}^2 &= 1 + x^2 \mathbf{B}^2 - 2xy AB^2 + y^2 B^2 \mathbf{A}^2 + w^2 \mathbf{A}^2 \mathbf{B}^2 - w^2 A^2 B^2 - \\ &\quad - z^2 \mathbf{A}^2 + 2zw BA^2 - w^2 A^2 \mathbf{B}^2 - y^2 \mathbf{A}^2 \mathbf{B}^2 + y^2 A^2 B^2 , \end{aligned}$$

that is

$$\begin{aligned} 1 + \mathbf{m}^2 - \mathbf{n}^2 &= 1 + x^2 \mathbf{B}^2 - z^2 \mathbf{A}^2 - 2xy AB^2 + 2zw BA^2 + \\ &\quad + y^2 [(B^2 - \mathbf{B}^2) \mathbf{A}^2 + A^2 B^2] - w^2 [(A^2 - \mathbf{A}^2) \mathbf{B}^2 + A^2 B^2] . \end{aligned}$$

The quadratic equation for parameters of the Mueller matrix takes the form

$$\begin{aligned} &x^2(A^2 - \mathbf{B}^2) + 2xy A(B^2 - \mathbf{A}^2) + y^2 [(\mathbf{A}^2 + \mathbf{B}^2 - B^2) \mathbf{A}^2 - A^2 B^2] = \\ &= z^2(B^2 - \mathbf{A}^2) + 2zw B(A^2 - \mathbf{B}^2) + w^2 [(\mathbf{A}^2 + \mathbf{B}^2 - A^2) \mathbf{B}^2 - A^2 B^2] + 1 . \end{aligned} \tag{56}$$

5. On defining 4-dimensional Mueller matrix from polarization measurements

As shown above, each polarization measurement

$$S_a \xrightarrow{L} S'_a \quad \text{or} \quad (A_a, B_a) \xrightarrow{L} (A'_a, B'_a)$$

allows to obtain the quadratic constraint on Mueller's characteristics of a polarization device

$$\begin{aligned} &x^2 (A^2 - \mathbf{B}^2) + 2xy A(B^2 - \mathbf{A}^2) + y^2 [(\mathbf{A}^2 + \mathbf{B}^2 - B^2) \mathbf{A}^2 - A^2 B^2] = \\ &= z^2 (B^2 - \mathbf{A}^2) + 2zw B(A^2 - \mathbf{B}^2) + w^2 [(\mathbf{A}^2 + \mathbf{B}^2 - A^2) \mathbf{B}^2 - A^2 B^2] + 1 ; \end{aligned} \tag{57}$$

the later has a 3-parametric set of solutions which describe all the possible Mueller matrices of the given optical device

$$\begin{aligned} n_0 &= x A - y \mathbf{A}^2 , & \mathbf{n} &= z \mathbf{A} - w AB + y \mathbf{A} \times \mathbf{B} , \\ m_0 &= -z B + w \mathbf{B}^2 , & \mathbf{m} &= x \mathbf{B} - y B \mathbf{A} + w \mathbf{A} \times \mathbf{B} . \end{aligned} \tag{58}$$

It is evident, that to fix Mueller matrix uniquely, one should perform several polarization tests. Let start with four ones – the problem to solve is formulate as a system of 4 equations

$$\begin{aligned}
& x^2 (A_1^2 - \mathbf{B}_1^2) + 2xy A_1(B_1^2 - \mathbf{A}_1^2) + y^2 [(\mathbf{A}_1^2 + \mathbf{B}_1^2 - B_1^2)\mathbf{A}_1^2 - A_1^2 B_1^2] = \\
& = z^2 (B_1^2 - \mathbf{A}_1^2) + 2zw B_1(A_1^2 - \mathbf{B}_1^2) + w^2 [(\mathbf{A}_1^2 + \mathbf{B}_1^2 - A_1^2)\mathbf{B}_1^2 - A_1^2 B_1^2] + 1 . \\
& x^2 (A_2^2 - \mathbf{B}_2^2) + 2xy A_2(B_2^2 - \mathbf{A}_2^2) + y^2 [(\mathbf{A}_2^2 + \mathbf{B}_2^2 - B_2^2)\mathbf{A}_2^2 - A_2^2 B_2^2] = \\
& = z^2 (B_2^2 - \mathbf{A}_2^2) + 2zw B_2(A_2^2 - \mathbf{B}_2^2) + w^2 [(\mathbf{A}_2^2 + \mathbf{B}_2^2 - A_2^2)\mathbf{B}_2^2 - A_2^2 B_2^2] + 1 . \\
& x^2 (A_3^2 - \mathbf{B}_3^2) + 2xy A_3(B_3^2 - \mathbf{A}_3^2) + y^2 [(\mathbf{A}_3^2 + \mathbf{B}_3^2 - B_3^2)\mathbf{A}_3^2 - A_3^2 B_3^2] = \\
& = z^2 (B_3^2 - \mathbf{A}_3^2) + 2zw B_3(A_3^2 - \mathbf{B}_3^2) + w^2 [(\mathbf{A}_3^2 + \mathbf{B}_3^2 - A_3^2)\mathbf{B}_3^2 - A_3^2 B_3^2] + 1 . \\
& x^2 (A_4^2 - \mathbf{B}_4^2) + 2xy A_4(B_4^2 - \mathbf{A}_4^2) + y^2 [(\mathbf{A}_4^2 + \mathbf{B}_4^2 - B_4^2)\mathbf{A}_4^2 - A_4^2 B_4^2] = \\
& = z^2 (B_4^2 - \mathbf{A}_4^2) + 2zw B_4(A_4^2 - \mathbf{B}_4^2) + w^2 [(\mathbf{A}_4^2 + \mathbf{B}_4^2 - A_4^2)\mathbf{B}_4^2 - A_4^2 B_4^2] + 1 . \tag{59}
\end{aligned}$$

It may be presented in a symbolical form as

$$\begin{aligned}
a_1 x^2 + 2b_1 xy + c_1 y^2 &= \alpha_1 z^2 + 2\beta_1 zw + \sigma_1 w^2 + 1 , \\
a_2 x^2 + 2b_2 xy + c_2 y^2 &= \alpha_2 z^2 + 2\beta_2 zw + \sigma_2 w^2 + 1 , \\
a_1 x^2 + 2b_3 xy + c_3 y^2 &= \alpha_3 z^2 + 2\beta_3 zw + \sigma_3 w^2 + 1 , \\
a_4 x^2 + 2b_4 xy + c_4 y^2 &= \alpha_4 z^2 + 2\beta_4 zw + \sigma_4 w^2 + 1 . \tag{60}
\end{aligned}$$

In general, this mathematical task should have a definite solution, though rather cumbersome one. Indeed, we could successively exclude the variables as follows

$$\begin{aligned}
(1) \quad & \implies x = x(y, z, w) , \\
(2) \quad & \implies y = y(z, w) , \quad x = x(y(z, w), z, w) = \bar{x}(z, w) , \\
(3) \quad & \implies z = z(w) , \quad (4) \quad \implies w = w(\dots) , \quad z = z(w(\dots)) .
\end{aligned}$$

However, there exist another and more beautiful way to solve the problem. Indeed, let us consider 6 independent polarization measurements – they provide us with 6 linear equations under 6 variables

$$\begin{aligned}
& x^2 , \quad y^2 , \quad 2xy , \quad z^2 , \quad w^2 , \quad 2zw ; \\
& a_1 x^2 + 2b_1 xy + c_1 y^2 - \alpha_1 z^2 - 2\beta_1 zw - \sigma_1 w^2 = +1 , \\
& a_2 x^2 + 2b_2 xy + c_2 y^2 - \alpha_2 z^2 - 2\beta_2 zw - \sigma_2 w^2 = +1 , \\
& a_1 x^2 + 2b_3 xy + c_3 y^2 - \alpha_3 z^2 - 2\beta_3 zw - \sigma_3 w^2 = +1 , \\
& a_4 x^2 + 2b_4 xy + c_4 y^2 - \alpha_4 z^2 - 2\beta_4 zw - \sigma_4 w^2 = +1 , \\
& a_5 x^2 + 2b_5 xy + c_5 y^2 - \alpha_5 z^2 - 2\beta_5 zw - \sigma_5 w^2 = +1 , \\
& a_6 x^2 + 2b_6 xy + c_6 y^2 - \alpha_6 z^2 - 2\beta_6 zw - \sigma_6 w^2 = +1 . \tag{61}
\end{aligned}$$

By physical reasons, we can presuppose existence of a unique solution of the task. This is given by Kramer's rule

$$\begin{aligned}
x^2 &= \frac{\Delta_{x^2}}{\Delta} , \quad y^2 = \frac{\Delta_{y^2}}{\Delta} , \quad 2xy = \frac{\Delta_{2xy}}{\Delta} , \\
z^2 &= \frac{\Delta_{z^2}}{\Delta} , \quad w^2 = \frac{\Delta_{w^2}}{\Delta} , \quad 2zw = \frac{\Delta_{2zw}}{\Delta} , \tag{62}
\end{aligned}$$

from whence it follows (evidently, arising subtleties with \pm should be examined additionally)

$$\begin{aligned} x + y &= \sqrt{\frac{\Delta_{x^2} + \Delta_{y^2} + \Delta_{2xy}}{\Delta}}, & x - y &= \sqrt{\frac{\Delta_{x^2} + \Delta_{y^2} - \Delta_{2xy}}{\Delta}}, \\ z + w &= \sqrt{\frac{\Delta_{z^2} + \Delta_{w^2} + \Delta_{2zw}}{\Delta}}, & z - w &= \sqrt{\frac{\Delta_{z^2} + \Delta_{w^2} - \Delta_{2zw}}{\Delta}}, \end{aligned} \quad (63)$$

Recall (see (51)) that Muller's matrices are defined by k -parameter

$$\begin{aligned} k_0 &= (xA - izB) - (y\mathbf{A}^2 - iw\mathbf{B}^2), \\ \mathbf{k} &= -(yB + iz)\mathbf{A} + (x + iwA)\mathbf{B} + (w - iy)\mathbf{A} \times \mathbf{B}; \end{aligned}$$

evidently, any orthogonal Lorentz matrix cannot distinguish between $(+k_0, +\mathbf{k})$ and $(-k_0, -\mathbf{k})$.

We may employ the same method in non-relativistic case as well. See (55); with the notation $z = \nu, y = N$ we have

$$n_0^2 + \mathbf{n}^2 = 1, \quad n_0 = y\mathbf{A}^2, \quad \mathbf{n} = z\mathbf{A} + y\mathbf{A} \times \mathbf{B}. \quad (64)$$

Note that because

$$\mathbf{A}^2 = (\mathbf{S} + \mathbf{S}')^2 = \mathbf{S}^2 + \mathbf{S}'^2 + 2\mathbf{S}\mathbf{S}' = 2(S^2 + \mathbf{S}\mathbf{S}'), \quad \mathbf{A} \times \mathbf{B} = 2\mathbf{S} \times \mathbf{S}',$$

eqs. (64) are equivalent to

$$n_0 = 2y(S^2 + \mathbf{S}\mathbf{S}'), \quad \mathbf{n} = z\mathbf{A} + 2y\mathbf{S} \times \mathbf{S}'. \quad (65)$$

and thereby coincide with (9)

$$n_0 = \beta(S^2 + \mathbf{S}\mathbf{S}'), \quad \mathbf{n} = \alpha(\mathbf{S} + \mathbf{S}') + \beta\mathbf{S} \times \mathbf{S}'. \quad (66)$$

In this notation two independent polarization test provide us with a linear system

$$\begin{aligned} y^2[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2] + z^2\mathbf{A}_1^2 &= 1, \\ y^2[\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2] + z^2\mathbf{A}_2^2 &= 1, \end{aligned} \quad (67)$$

its solution is

$$\begin{aligned} y^2 &= \frac{(\mathbf{A}_1\mathbf{B}_1)^2 - (\mathbf{A}_2\mathbf{B}_2)^2}{[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]\mathbf{A}_2^2 - [\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2]\mathbf{A}_1^2}, \\ z^2 &= \frac{[\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2] - [\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]}{[\mathbf{A}_1^2(\mathbf{A}_1^2 + \mathbf{B}_1^2) - (\mathbf{A}_1\mathbf{B}_1)^2]\mathbf{A}_2^2 - [\mathbf{A}_2^2(\mathbf{A}_2^2 + \mathbf{B}_2^2) - (\mathbf{A}_2\mathbf{B}_2)^2]\mathbf{A}_1^2}. \end{aligned} \quad (68)$$

6. On diagonalizing the transitivity equation

The transitivity equation $LS = S'$ led us to a 3-surface in 4-parametric space

$$\begin{aligned} x^2(A^2 - \mathbf{B}^2) + 2xyA(B^2 - \mathbf{A}^2) + y^2[(\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2B^2] - \\ - z^2(B^2 - \mathbf{A}^2) - 2zwB(A^2 - \mathbf{B}^2) - w^2[(\mathbf{A}^2 + \mathbf{B}^2 - A^2)\mathbf{B}^2 - A^2B^2] = 1, \end{aligned} \quad (69)$$

or in symbolical form

$$ax^2 + 2bxy + cy^2 - \alpha z^2 - 2\beta zw - \sigma w^2 = +1 . \quad (70)$$

Let us examine the possibility to transform an elementary quadratic form to a diagonal form by means of 3-rotation in 2-plane

$$\begin{aligned} ax^2 + 2bxy + cy^2 &= FX^2 + GY^2 , \\ x &= \cos \phi X + \sin \phi Y , \quad y = -\sin \phi X + \cos \phi Y . \end{aligned} \quad (71)$$

Eqs. (71) yield

$$\begin{aligned} a(\cos \phi X + \sin \phi Y)^2 + 2b(\cos \phi X + \sin \phi Y)(-\sin \phi X + \cos \phi Y) + \\ + c(-\sin \phi X + \cos \phi Y)^2 = FX^2 + GY^2 \implies \\ a(2XY \sin \phi \cos \phi + X^2 \cos^2 \phi + Y^2 \sin^2 \phi) + \\ + 2b[(Y^2 - X^2) \sin \phi \cos \phi + XY(\cos^2 \phi - \sin^2 \phi)] + \\ + c(-2XY \sin \phi \cos \phi + X^2 \sin^2 \phi + Y^2 \cos^2 \phi) = FX^2 + GY^2 . \end{aligned} \quad (72)$$

So we have three equations

$$\begin{aligned} X^2 : \quad a \cos^2 \phi - 2b \sin \phi \cos \phi + c \sin^2 \phi &= F , \\ Y^2 : \quad a \sin^2 \phi + 2b \sin \phi \cos \phi + c \cos^2 \phi &= G , \\ 2XY : \quad a \sin \phi \cos \phi + b(\cos^2 \phi - \sin^2 \phi) - c \sin \phi \cos \phi &= 0 . \end{aligned}$$

With the help of the variable 2ϕ , these are written as

$$\begin{aligned} a \frac{\cos 2\phi + 1}{2} - b \sin 2\phi + c \frac{1 - \cos 2\phi}{2} &= F , \\ a \frac{1 - \cos 2\phi}{2} + b \sin 2\phi + c \frac{\cos 2\phi + 1}{2} &= G , \\ \frac{a - c}{2} \sin 2\phi + b \cos 2\phi &= 0 . \end{aligned} \quad (73)$$

This results in

$$\sin 2\phi = \frac{2b}{\sqrt{(c-a)^2 + 4b^2}} , \quad \cos 2\phi = \frac{c-a}{\sqrt{(c-a)^2 + 4b^2}} ; \quad (74)$$

and

$$\begin{aligned} F &= \frac{a+c}{2} + \frac{a-c}{2} \cos 2\phi - b \sin 2\phi = \frac{a+c}{2} - \frac{\sqrt{(a-c)^2 + 4b^2}}{2} , \\ G &= \frac{a+c}{2} - \frac{a-c}{2} \cos 2\phi + b \sin 2\phi = \frac{a+c}{2} + \frac{\sqrt{(a-c)^2 + 4b^2}}{2} . \end{aligned} \quad (75)$$

In the same manner, the second quadratic form is considered

$$\begin{aligned} -\alpha z^2 - 2\beta zw - \sigma w^2 &= \Delta Z^2 + \Gamma W^2 \\ z &= \cos \rho Z + \sin \rho W , \quad w = -\sin \rho Z + \cos \rho W . \end{aligned} \quad (76)$$

For 2ρ we get

$$\sin 2\rho = \frac{2\beta}{\sqrt{(\sigma - \alpha)^2 + 4\beta^2}}, \quad \cos 2\rho = \frac{\sigma - \alpha}{\sqrt{(\sigma - \alpha)^2 + 4\beta^2}}; \quad (77)$$

$$\begin{aligned} \Delta &= \frac{\alpha + \sigma}{2} - \frac{\sqrt{(\alpha - \sigma)^2 + 4\beta^2}}{2}; \\ \Gamma &= \frac{\alpha + \sigma}{2} + \frac{\sqrt{(\alpha - \sigma)^2 + 4\beta^2}}{2}. \end{aligned} \quad (78)$$

For instance, conditions at which F and G are positive, and Δ, Γ are negative, are formulated in the form

$$(F, G, \Delta, \Gamma) \sim (+, +, -, -),$$

$$\begin{aligned} a > 0, \quad c > 0, \quad a + c > +\sqrt{(a - c)^2 + 4b^2} > 0 &\implies ac > b^2. \\ \alpha < 0, \quad \sigma < 0, \quad \alpha + \sigma < -\sqrt{(\alpha - \sigma)^2 + 4\beta^2} &\implies \alpha\sigma > \beta^2. \end{aligned} \quad (79)$$

When specifying expressions for $a, b, c, \alpha, \beta, \sigma$ we should distinguish between a partly and completely polarized light. In the case of a partly polarized and completely polarized light we have respectively

$$\begin{aligned} S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}'^2 = 0, \quad S_0 = +|\mathbf{S}|, \\ S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}'^2 > 0, \quad S_0 > |\mathbf{S}|. \end{aligned}$$

For the main invariant let us use the notation $S_0^2 - \mathbf{S}^2 = S_0'^2 - \mathbf{S}'^2 = \Sigma^2$.

Expression for a, b, α, β are given by

$$\begin{aligned} a &= (S_0 + S_0')^2 - (\mathbf{S} - \mathbf{S}')^2 = 2\Sigma^2 + 2(S_0S_0' + \mathbf{S}\mathbf{S}'), \\ \frac{b}{A} &= (S_0 - S_0')^2 - (\mathbf{S} + \mathbf{S}')^2 = 2\Sigma^2 - 2(S_0S_0' + \mathbf{S}\mathbf{S}'), \\ \alpha &= (S_0 - S_0')^2 - (\mathbf{S} + \mathbf{S}')^2 = 2\Sigma^2 - 2(S_0S_0' + \mathbf{S}\mathbf{S}'), \\ \frac{\beta}{B} &= (S_0 + S_0')^2 - (\mathbf{S} - \mathbf{S}')^2 = 2\Sigma^2 + 2(S_0S_0' + \mathbf{S}\mathbf{S}'). \end{aligned} \quad (80)$$

they become simpler for a completely polarized light

$$\begin{aligned} a_{polar} &= +2(S_0S_0' + \mathbf{S}\mathbf{S}') > 0, \quad \frac{b_{polar}}{A} = -2(S_0S_0' + \mathbf{S}\mathbf{S}') < 0, \\ \alpha_{polar} &= -2(S_0S_0' + \mathbf{S}\mathbf{S}') < 0, \quad \frac{\beta_{polar}}{B} = +2(S_0S_0' + \mathbf{S}\mathbf{S}') > 0. \end{aligned} \quad (81)$$

Let us specify $c = (\mathbf{A}^2 + \mathbf{B}^2 - B^2)\mathbf{A}^2 - A^2B^2$; accounting for

$$\begin{aligned} \mathbf{A}^2 + \mathbf{B}^2 - B^2 &= (\mathbf{S} + \mathbf{S}')^2 + (\mathbf{S} - \mathbf{S}')^2 - (S_0 - S_0')^2 = -4\Sigma^2 + (S_0 + S_0')^2, \\ \mathbf{A}^2 &= (\mathbf{S} + \mathbf{S}')^2, \quad A^2B^2 = (S_0 + S_0')^2(S_0 - S_0')^2 \end{aligned}$$

we get

$$c = [-4\Sigma^2 + (S_0 + S'_0)^2](\mathbf{S} + \mathbf{S}')^2 - (S_0 + S'_0)^2(S_0 - S'_0)^2 ,$$

$$c_{polar} = 2(S_0 + S'_0)^2 (S_0 S_0 + \mathbf{S}\mathbf{S}') . \quad (82)$$

In the same mater, for $\sigma = \sigma = (\mathbf{B}^2 + \mathbf{A}^2 - A^2)\mathbf{B}^2 - B^2 A^2$ with relations

$$\mathbf{B}^2 + \mathbf{A}^2 - A^2 = (\mathbf{S} - \mathbf{S}')^2 + (\mathbf{S} + \mathbf{S}')^2 - (S_0 + S'_0)^2 = -4\Sigma^2 + (S_0 - S'_0)^2 ,$$

$$\mathbf{B}^2 = (\mathbf{S} - \mathbf{S}')^2 , \quad B^2 A^2 = (S_0 - S'_0)^2 (S_0 + S'_0)^2$$

we obtain

$$\sigma = [-4\Sigma^2 + (S_0 - S'_0)^2](\mathbf{S} - \mathbf{S}')^2 - (S_0 - S'_0)^2(S_0 + S'_0)^2 ,$$

$$\sigma_{polar} = -2(S_0 - S'_0)^2 (S_0 S_0 + \mathbf{S}\mathbf{S}') \quad (83)$$

7. On the Lorentz little group for a partly polarized light

In the context of polarization optics, some interest may have the known problem of the little Lorentz group. What is the majority of Mueller matrices leaving invariant a given Stokes 4-vector. The problem is reduced to

$$L_b^a(k, \bar{k}^*) S_a = +S_b , \quad S^a S_a = \text{inv} > 0 ; \quad (84)$$

with the use of a factorized form $L = A A^* = A^* A$, the previous equations are

$$A S = (A^*)^{-1} S \quad \implies \quad [A - (A^*)^{-1}] S = 0 , \quad (85)$$

$$A = \begin{vmatrix} k_0 & -k_1 & -k_2 & -k_3 \\ -k_1 & k_0 & -ik_3 & ik_2 \\ -k_2 & ik_3 & k_0 & -ik_1 \\ -k_3 & -ik_2 & ik_1 & k_0 \end{vmatrix} , \quad (A^*)^{-1} = \begin{vmatrix} k_0^* & k_1^* & k_2^* & k_3^* \\ k_1^* & k_0^* & -ik_3^* & ik_2^* \\ k_2^* & ik_3^* & k_0^* & -ik_1^* \\ k_3^* & -ik_2^* & ik_1^* & k_0^* \end{vmatrix} .$$

So we arrive at

$$\begin{vmatrix} (k_0 - k_0^*) & -(k_1 + k_1^*) & -(k_2 + k_2^*) & -(k_3 + k_3^*) \\ -(k_1 + k_1^*) & (k_0 - k_0^*) & -i(k_3 - k_3^*) & i(k_2 - k_2^*) \\ -(k_2 + k_2^*) & i(k_3 - k_3^*) & (k_0 - k_0^*) & -i(k_1 - k_1^*) \\ -(k_3 + k_3^*) & -i(k_2 - k_2^*) & i(k_1 - k_1^*) & (k_0 - k_0^*) \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0 \quad (86)$$

which with notation $k_0 = n_0 + im_0$, $k_j = -in_j + m_j$ reads

$$\begin{vmatrix} im_0 & -m_1 & -m_2 & -m_3 \\ -m_1 & im_0 & -n_3 & n_2 \\ -m_2 & n_3 & im_0 & -n_1 \\ -m_3 & -n_2 & n_1 & im_0 \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0 \quad (87)$$

Note that imposing restrictions $m_0 = 0, m_j = 0$, we obtain a more simple equation

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -n_3 & n_2 \\ 0 & n_3 & 0 & -n_1 \\ 0 & -n_2 & n_1 & 0 \end{vmatrix} \begin{vmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{vmatrix} = 0 \quad \implies \quad \mathbf{n} = \frac{\mathbf{S}}{S} \quad (88)$$

which describes a 1-parametric group of 3-rotations $O(\phi, \mathbf{n})$ about the axis $\mathbf{S} = S\mathbf{n}$. In general case, eq. (87) can be presented in the vector form

$$im_0S_0 - \mathbf{m}\mathbf{S} = 0, \quad -\mathbf{m}S_0 + im_0\mathbf{S} + \mathbf{n} \times \mathbf{S} = 0. \quad (89)$$

To have solutions in real variables, we must require $m_0 = 0$. Therefore, an expression for \mathbf{m} is

$$\mathbf{m} = \frac{\mathbf{n} \times \mathbf{S}}{S_0} = \mathbf{n} \times \mathbf{p}. \quad (90)$$

Thus, solution for the problem of little Lorentz group is (first it was obtained by Wigner [...])

$$\begin{aligned} L_b^a(k, \bar{k}^*) S_a &= +S_b, & S^a S_a &= \text{inv} > 0; \\ k_0 &= n_0 + i0, & \mathbf{k} &= -i \mathbf{n} + \mathbf{n} \times \mathbf{p}. \end{aligned} \quad (91)$$

Explicitly, additional condition for parameters looks

$$k_0^2 - \mathbf{k}^2 = 1 \quad \implies \quad n_0^2 + \mathbf{n}^2(1 - \mathbf{p}^2) + (\mathbf{n}\mathbf{p})^2 = 1. \quad (92)$$

This relationship determines a 3-parametric majority of jueller matrices leaving invariant the polarization vector $S_a = (S_0, S_0 p_i)$ of the partly polarized light. As known, this set of transformations consists of a group isomorphic to $SU(2)$.

8. On the Lorentz little group for a completely polarized light

Analogous problem for a completely polarized light looks much the same

$$L_b^a(k, \bar{k}^*) S_a = +S_b, \quad S^a S_a = 0;$$

we again have equations

$$im_0S_0 - \mathbf{m}\mathbf{S} = 0, \quad -\mathbf{m}S_0 + im_0\mathbf{S} + \mathbf{n} \times \mathbf{S} = 0,$$

in which restriction $m_0 = 0$ must hold. Solution looks as follows

$$\begin{aligned} L_b^a(k, \bar{k}^*) S_a &= +S_b, & S^a S_a &= 0; \\ k_0 &= n_0 + i0, & \mathbf{k} &= -i \mathbf{n} + \mathbf{n} \times \mathbf{p}, & \mathbf{p}^2 &= 1. \end{aligned} \quad (93)$$

The difference arises due to the relation $\mathbf{p}^2 = 1$,

$$k_0^2 - \mathbf{k}^2 = 1 \quad \implies \quad n_0^2 + (\mathbf{n}\mathbf{p})^2 = 1. \quad (94)$$

This relationship determines a 3-parametric majority of Mueller matrices leaving invariant a given isotropic Stokes 4-vector $S_a = (S_0, S_0 p_i)$, $\mathbf{p}^2 = 1$.

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References

- [1] G.G. Stokes. On the composition and resolution of streams of polarized light from different sources. Trans. Cambridge Phil. Soc. **9**, 399 (1952).
- [2] R.C. Jones. New calculus for the treatment of optical systems. I. Description and discussion of the calculus. J. Opt. Soc. Amer. **31**, 488-493 (1941).
- [3] H. Mueller. Memorandum on the polarization optics of the photo-elastic shutter. Reporn No 2 of the OSRD project OEMsr576, Nov. 15 (1943).
- [4] F.I. Fedorov. Optics of anisotropic medias. Minsk, 1958 (in Russian).
- [5] M. Born, E. Wolf. Principles of Optics, 6th Ed. (Pergamon, Oxford, 1980); the first edition of this book was published in 1959.
- [6] G.P. Parent, P. Roman. On the matrix formulation of the theory of partial polarization in terms of observables. Nuovo Cimento. **15**, 370-388 (1960).
- [7] W.A. Shurcliff. Polarized light, production and use. (Harvard Univ. Press, Cambridge, MA, 1962; Russian translation in 1965, Mir, Moskow.
- [8] R. Barakat. Theory of the coherency matrix for light of arbitrary bandwidth. J. Opt. Soc. Am. **53**, 317-323 (1963).
- [9] R.W. Schmieder. Stokes-algebra formalism. J. Opt. Soc. Am. **59**, 297-302 (1969).
- [10] H. Takenaka. A unified formalism for polarization optics by using group theory. Nouvelle Revue d'Optique. **4**, 37-41 (1973).
- [11] A. Gerrard, J.M. Burch. Introduction to matrix methods in optics. 1975; translation to Russian, 1978.
- [12] W. Swindell. Polarized Light, Dowden, Hutchinson, and Ross, Inc., Stroudsburg, PA, 1975.
- [13] R.C. Jones. Polarized light. W. Swindell ed. Stroudsburg PA. 1975
- [14] F.I. Fedorov. The theory of hyrotropy. Minsk, 1976.
- [15] R. A. M. Azzam, I. Bashara. Ellipsometry and polarized light. North-Holland, Amsterdam, 1977.
- [16] R.M.A. Azzam. Propagation of partially polarized light through anisotropic media with or without depolarization: a differential 4×4 matrix calculus. J. Opt. Soc. Am. **68**, 1756-1767 (1978).
- [17] D. Han, Y.S. Kim, D. Son. Photon spin as a rotation in gauge space Phys. Rev. D. **25**, No 2, 461 - 463 (1982).
- [18] M.V. Berry. Singularities in waves and rays. in R. Balian, M. Kleman, and J.-P. Poirier, editors, Les Houches Session XXV - Physics of Defects, North-Holland, 1981.

- [19] R. Barakat. Bilinear constraints between elements of the 4×4 Mueller-Jones transfer matrix of polarization theory. *Opt. Commun.* **38**, 159-161 (1981).
- [20] R. Simon. The connection between Mueller and Jones matrices of polarization optics. *Opt. Commun.* **42**, 293-297 (1982).
- [21] P. Yeh. Extended Jones matrix method. *J. Opt. Soc. Am.* **72**, 507 (1982).
- [22] J.F. Nye. Lines of circular polarization in electromagnetic wave fields. *Proc. R. Soc. London. A.* **389**, 279-290 (1983).
- [23] E.C.G. Sudarshan, R. Simon, N. Makunda. Paraxial-wave optics and relativistic front description. I. The scalar theory. *Phys. Rev. A.* **28**, 2921-2932 (1983); N. Makunda, R. Simon, E.C.G. Sudarshan. Paraxial-wave optics and relativistic front description. II. The vector theory. *Phys. Rev. A.* **28**, 2933-2942 (1983).
- [24] J.J. Gil, E. Bernabeau. A depolarization criterion in Mueller matrices. *Optica Acta.* **32**, 259-261 (1985).
- [25] S.R. Cloude. Group theory and polarization algebra. *Optik.* **75**, 26-36 (1986).
- [26] R. Simon. Mueller matrices and depolarization criteria. *J. Mod. Optics.* **34**, 569-575 (1987).
- [27] R. Simon. Non-depolarizing systems and degree of polarization. *Opt. Commun.* **77**, 349-354 (1990).
- [28] J.F. Nye and J.V. Hajnal. The wave structure of monochromatic electromagnetic radiation. *Proc. R. Soc. London. A.* **409**, 21-36 (1987).
- [29] A.B. Kostinski, B. James, W.M. Boerner. Optimal reception of partially polarized waves. *J. Opt. Soc. Am. A.* **5**, 58-64 (1988).
- [30] D. Han, Y.S. Kim, D. Son. Unitary transformations of photon polarization vectors. *Phys. Rev. D.* **31**, No 2, 328 - 330 (1985).
- [31] K. Kim, L. Mandel, E. Wolf. Relationship between Jones and Mueller matrices for random media. *J. Opt. Soc. A.* **4**, 433-437 (1987).
- [32] D. Han, Y.S. Kim, Marilyn E. Noz. Uncertainty relations for light waves and the concept of photons. *Phys. Rev. A.* **35**, No 4, 1682 - 1691 (1987).
- [33] K. Kim, L. Mandel, E. Wolf. Relationship between Jones and Mueller matrices for random media. *J. Opt. Soc. Amer. A.* **4**, 433-437 (1987).
- [34] R.Y. Chiao, T.F. Jordan. Lorentz-group Berry phases in squeezed light. *Phys. Lett. A.* **132**, 77-81 (1988).
- [35] Y.S. Kim, E.P. Wigner. Covariant phase-space representation for localized light waves. *Phys. Rev. A.* **36**, No 3, 1293 - 1297 (1987).
- [36] D. Han, Y.S. Kim. Special relativity and interferometers. *Phys. Rev. A.* **37**, No 11, 4494 - 4496 (1988).

- [37] S. R. Cloude. Conditions for the physical realisability of matrix operators in polarimetry. in: Polarization Considerations for Optical Systems II, R.A. Chipman ed., Proc. Soc. Photo-Opt. Instrum. Eng. **1166**, 36-43 (1989).
- [38] D. Han, E.E. Hardekop, Y.S. Kim. Thomas precession and squeezed states of light. Phys. Rev. A. **39**, No 3, 1269 - 1276 (1989).
- [39] M. Kitano, T. Yabuzaki. Observation of Lorentz-group Berry phases in polarization optics. Phys. Lett. A. **142**, 321-325 (1989).
- [40] S.R. Cloude. Lie groups in electromagnetic wave propagation and scattering. Journal of Electromagnetic Waves and Applications. **6**, 947-974 (1992).
- [41] M. Sanjay Kumar, R. Simon. Characterization of Mueller matrices in polarization optics. Optics Commun. **88**, 464-470 (1992).
- [42] A. B. Kostinski. Depolarization criterion for incoherent scattering. Appl. Optics. **31**, 506-3508 (1992).
- [43] Y.S. Kim, L.H. Yeh. For an $O(2, 1)$ -based approach to light waves in non-vacuum media. J. Math. Phys. **33**, 1237 (1992).
- [44] C.V.M. van der Mee, J.W. Hovenier. Structure of matrices transforming Stokes parameters. J. Math. Phys. **33**, 3574-3584 (1992).
- [45] P. Pellat-Finet, M. Buasset. What is common to both polarization optics and relativistic kinematics? Optik (Stuttgart) **90**, 101-106 (1992).
- [46] V.N. Snopko. Polarization characteristics of optical radiation and methods of their measurement. Minsk, 1992 (in Russian).
- [47] C.R. Givens, A.B. Kostinski. A simple necessary and sufficient criterion on physically realizable Mueller matrices. J. Mod. Opt. **40**, 471-481 (1993).
- [48] C.V.M. van der Mee. An eigenvalue criterion for matrices transforming Stokes parameters. J. Math. Phys. **34**, 5072-5088 (1993).
- [49] T. Opatrný, J. Perina. Non-image-forming polarization optical devices and Lorentz transformation – an analogy. Phys. Lett. A. **181**, 199-202 (1993).
- [50] J.L. Pezzanini. Mueller matrix imaging. Ph.D. No 9415102. 269 pages, The University of Alabama in Huntsville, 1993.
- [51] T. Opatrný, J. Perina. Non-image-forming polarization optical devices and Lorentz transformations - analogy. Phys. Lett. A. **181**, 199-202 (1993).
- [52] U. Leonhardt. Lorentz-group Berry phases via two-mode squeezing. Optics Commun. **104**, 81-84 (1993).
- [53] U. Leonhardt. Quantum statistics of a lossless beam splitter: $SU(2)$ symmetry in phase-space. Phys. Rev. A. **48**, 3265-3277 (1993).

- [54] U. Leonhardt. Quantum statistics of a $SU(1,1)$ interferometer. *Phys. Rev. A.* **49**, 1231-1242 (1994).
- [55] Shih-Yau Lu, Russell A. Chipman. Homogeneous and inhomogeneous Jones matrices. *J. Opt. Soc. Amer. A.* **11**, No 2, 766-773 (1994).
- [56] D.G.M. Anderson, R. Barakat. Necessary and sufficient conditions for a Mueller matrix to be derivable from a Jones matrix. *J. Opt. Soc. Am. A.* **11**, No 8, 2305-2319 (1994).
- [57] R. Sridhar, R. Simon. Normal form for Mueller matrices in polarization optics. *J. Mod. Optics.* **41**, 1903-1915 (1994).
- [58] C.S. Brown, A.E. Bak. Unified formalism for polarization optics with application to polarimetry on a twisted optical fiber. *Opt. Eng.* **34**, 1625-1635 (1995).
- [59] C. Brosseau. Evolution of the Stokes parameters in optically anisotropic media. *Opt. Lett.* **20**, 1221-1223 (1995).
- [60] J. Lehner, U. Leonhardt, H. Paul. Unpolarized light: Classical and quantum states. *Phys. Rev. A.* **53**, 2727-2735 (1996).
- [61] D. Han, Y.S. Kim, M.E. Noz. Polarization optics and bilinear representation of the Lorentz group. *Phys. Lett. A.* **219**, 26-32 (1996).
- [62] Arvind, N. Mukunda. Relativistic operator description of photon polarization [quant-ph/9612024].
- [63] Shih-Yau Lu, Russell A. Chipman. Interpretation of Mueller matrices based on polar decomposition. *J. Opt. Soc. Amer. A.* **13**, No 5, 1106-1116 (1996).
- [64] M. Gutierrez, J.C. Minano, C. Vega, P. Benitez. Application of Lorentz geometry to non-imaging optics: New three-dimensional ideal contractors. *J. Opt. Soc. Am. A.* **13**, 532-542 (1996).
- [65] D. Han, Y.S. Kim, M.E. Noz. Jones-matrix formalism as a representation of the Lorentz group. *J. Opt. Soc. Am. A.* **14**, No 9, 2290-2298 (1997) [physics/9703032].
- [66] D. Han, Y.S. Kim, M.E. Noz. Stokes parameters as a Minkowskian four-vector. *Phys. Rev. E.* **56**, No 5, 6065-6076 (1997) [physics/9707016].
- [67] Dr. Sudha. Some Algebraic Aspects of Relativity and Polarization Optics. Ph.D. Thesis, submitted to the University of Mysore, 1998.
- [68] H. Kuratsuji, S. Kakigi. Maxwell-Schrödinger equation for polarized light and evolution of the Stokes parameters. *Phys. Rev. Lett.* **80**, 1888-1891 (1998).
- [69] A. V. Gopala Rao, K. S. Mallesh, Sudha. On the algebraic characterization of a Mueller matrix in polarization optics I. Identifying a Mueller matrix from its N matrix. *J. Mod. Optics.* **45**, 955-987 (1998).
- [70] C. Brosseau. Fundamentals of polarized light. (John Wiley & Sons, 1998).

- [71] D. Han, Y.S. Kim, Marilyn E. Noz. Wigner rotations and Iwasawa decompositions in polarization optics. Phys. Rev. E. **60**, No 1, 1036 - 1041 (1999).
- [72] L. Allen, M.J. Padgett, M. Babiker. The orbital angular momentum of light. Prog. Opt. **39**, 291-372 (1999).
- [73] D. Han., Y.S. Kim, Marilyn E. Noz. Department of Radiology, New York University, Space-time symmetry transformations of elementary particles realized in optics laboratories [hep-th/9904130].
- [74] Sudha, A.V.Gopala Rao. Polarization elements – a group theoretical study [physics/0007079].
- [75] D. Han, Y.S. Kim, E. Noz. Interferometers and decoherence matrices. Phys. Rev. E. **61**, no 5, 5907 - 5913 (2000) [quant-ph/0003044].
- [76] E. Georgieva, Y.S. Kim. Iwasawa effects in multilayer optics. Phys. Rev. E. **64**, no 2, 26602 (6 pages) (2001) [physics/0103096].
- [77] S. Bařkal, Y.S. Kim. Wigner rotations in laser cavities. Phys. Rev. E. **66**, no 2, 026604 (6 pages) (2002) [math-ph/0108028].
- [78] S. Bařkal, Y.S. Kim. Shear representations of beam transfer matrices Phys. Rev. E. **63**, 56606 (6 pages) (2001).
- [79] M.S. Soskin, M.V. Vasnetsov. Singular optics. Prog. Opt. **42**, 219-276 (2001).
- [80] I. Freund. Polarization singularity indices in Gaussian laser beams. Opt. Commun. **201**, 251-270 (2002).
- [81] M.R. Dennis. Polarization singularities in paraxial vector fields: morphology and statistics. Opt. Commun. **213**, 201-221 (2002).
- [82] U. Leonhardt. Quantum physics of simple optical instruments. Rep. Prog. Phys. **66**, 1207-1249 (2003).
- [83] Sibel Bařkal, Elena Georgieva, Y.S. Kim. Wigner’s new physics frontier: Physics of two-by-two matrices, including the Lorentz group and optical instruments [math-ph/0310068].
- [84] S. Bařkal, Y.S. Kim. Lens optics as an optical computer for group contractions. Phys. Rev. E. **67**,no 5, 056601 (8 pages) (2003) [math-ph/0210056].
- [85] Sibel Bařkal, Elena Georgieva, Y.S. Kim. Lorentz group in ray optics [quant-ph/0401098].
- [86] Elena Georgieva, Y.S. Kim. Slide-rule-like property of Wigner’s little groups and cyclic S matrices for multilayer optics. Phys. Rev. E. **68**, no 2, 026606 (7 pages) (2003) [math-ph/0303019].
- [87] D. Han, Y.S. Kim, M.E. Noz. Wigner rotations and Iwasawa decompositions in polarization optics [quant-ph/0408181].

- [88] S. E. Ahnert, M.C. Payne. General implementation of all possible positive-operator-value measurements of single photon polarization states. *Phys. Rev. A*. **71**, 012330-33, (2005).
- [89] A. Aiello, J.P. Woerdman. Notes on Polarization Measurements [quant-ph/0503124].
- [90] Sibel Bařkal, Elena Georgieva, Y.S. Kim. Lorentz Group applicable to Finite Crystals [math-ph/0607035].
- [91] R. Botet, H. Kuratsuji, R. Seto. Novel aspects of evolution of the Stokes parameters for an electromagnetic wave in anisotropic media. *Prog. Theor. Phys.* **116**, 285-294 (2006).
- [92] A. Aiello, G. Puentes, J.P. Woerdman Linear optics and quantum maps [quant-ph/0611179].
- [93] A. Aiello, J.P. Woerdman. Linear Algebra for Mueller Calculus [math-ph/0412061].
- [94] Sudha, A.V. Gopala Rao, A.R. Usha Devi, A.K. Rajagopa. A POVM view of the ensemble approach to polarization optics. [arXiv:0704.0147v2].
- [95] Sudha, A.V. Gopala Rao, A.K. Rajagopal. A Unified ensemble approach to classical polarization optics. [arXiv:0704.0147].
- [96] Y.A. Kravtsov, B. Bieg, K.Y. Bliokh. Stokes-vector evolution in a weakly anisotropic inhomogeneous medium. *J. Opt. Soc. Am. A* **24**, 3388-3396 (2007).
- [97] S. Bařkal, Elena Georgieva, Y.S. Kim. Diagonalization of $Sp(2)$ matrices. [math-ph/0701028].
- [98] Konstantin Yu. Bliokh, Avi Niv, Vladimir Kleiner, Erez Hasman. Singular polarimetry: Evolution of polarization singularities in electromagnetic waves propagating in a weakly anisotropic medium. Vol. 16, No. 2 OPTICS EXPRESS 697.
- [99] Bogush A. A., Dlugunovich V. A., Zhukovich S. Ya., Kurochin Yu. A., Snopko V. N. Bi-quaternious and Mueller matrices. *Doklady of the National Academy of Sciences of Belarus*. Vol. 51, No 5. 71–76 (2007).
- [100] Bogush A.A. Mueller matrices in polarization optics. *Proc. of the Natl. Academy of Sciences of Belarus, Ser. Phys.-Math. Sci.*, No 2, P. 96 – 102 (2008).
- [101] V.M. Red'kov. Maxwell Equations in Media, Group Theory and Polarization of the Light. 73 pages, arxiv/0906.2482.
- [102] F.I. Fedorov, The Lorentz group, Moscow, 1979.
- [103] A.V. Berezin, Yu.A. Kurochkin, E.A. Tolkachev, Quaternions in relativistic physics, Minsk, 1989.
- [104] V.M. Red'kov. Fields in Riemannian space and the Lorentz group. Publishing House "Belarusian Science", Minsk, 496 pages, (2009).
- [105] Einstein A., Mayer W. Semivektoren und Spinoren. *Sitz. Ber. Preuss. Akad. Wiss. Berlin. Phys.-Math. Kl.* 522-550 (1932); Die Diracgleichungen für Semivektoren. *Proc. Akad. Wet. (Amsterdam)*. **36**, 497-516 (1933); Spaltung der Natürlichsten Feldgleichungen für Semivektoren in Spinor-Gleichungen von Diracschen Typus. *Proc. Akad. Wet. (Amsterdam)*. **36**, 615-619 (1933).